One-dimensional model of a Quantum nonlinear Harmonic Oscillator*

J. F. Cariñena

Departamento de Física Teórica, Facultad de Ciencias,
Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: jfc@posta.unizar.es

Manuel F. Rañada

Departamento de Física Teórica, Facultad de Ciencias,
Universidad de Zaragoza, 50009 Zaragoza, Spain
e-mail: mfran@unizar.es
and
Mariano Santander

Departamento de Física Teórica, Facultad de Ciencias
Universidad de Valladolid, 47011 Valladolid, Spain
e-mail: santander@fta.uva.es

February 1, 2008

Abstract

In this paper we study the quantization of the nonlinear oscillator introduced by Mathews and Lakshmanan. This system with position-dependent mass allows a natural quantization procedure and is shown to display shape invariance. Its energy spectrum is found by factorization. The linear harmonic oscillator appears as the $\lambda \to 0$ limit of this nonlinear oscillator, whose energy spectrum and eigenfunctions are compared to the linear ones.

MSC Classification: 34A34, 81U15, 34C15.

Key words: Nonlinear oscillator, position-dependent mass, exactly solvable systems.

1   Introduction

In a recent paper [1] we have analyzed a classical nonlinear oscillator which is a n-dimensional generalization of the one-dimensional system introduced previously

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*Presented to the 36 Symposium on Mathematical Physics, Toru ń 9–12 June 2004

by Mathews and Lakshmanan [2], [3], as a one-dimensional analogue of some models of quantum field theory [4], [5]. Such mechanical system was described by a Lagrangian

$$L = \frac{1}{2} \left(\frac{1}{1 + \lambda\,x^2} \right) (\dot{x}^2 - \alpha^2\,x^2)\,, \tag{1}$$

which represents a nonlinear oscillator-like with amplitude dependent frequency periodic solutions. Note that this system can also be considered as an oscillator with a position-dependent effective mass $m = (1 + \lambda\,x^2)^{-1}$ (see e.g. [6] and [7] and references therein).

The two-dimensional generalization studied in [1] was given by the Lagrangian

$$L(\lambda) = \frac{1}{2} \left(\frac{1}{1 + \lambda\,r^2} \right) \left[v_x^2 + v_y^2 + \lambda\,(x\,v_y - y\,v_x)^2 - \alpha^2\,r^2 \right], \quad r^2 = x^2 + y^2\,, \tag{2}$$

and it was shown to be not only integrable but super-integrable. This suggests that the corresponding quantum model should be exactly solvable, although one may expect to have some ordering ambiguities because of the x-dependence of the kinetic term.

We aim to study in this paper the one-dimensional quantum model using the well-known techniques of factorization and related operators (see e.g. [8] and [9] and references therein). This algebraic technique was started by Schrödinger in [10] and [11] and its interest has been increasing since the beginning of Supersymmetric Quantum Mechanics (see [12] for a review).

In more detail, the plan of the article is as follows: Section 2 is devoted to study the simplest $\alpha = 0$ case in the classical approach [1], both in the Lagrangian and the Hamiltonian formalism, and an infinitesimal symmetry as well as the invariant measure in $\mathbb{R}$ under such vector field are determined. Then, we proceed to introduce the quantum Hamiltonian describing this position-dependent free system. The factorization method for this kind of position-dependent mass [13] is developed in Section 3 and the specific example of the quantum nonlinear oscillator is studied in Section 4, where we prove that the problem has shape invariance. In Section 5 the spectrum of the quantum nonlinear oscillator is found by using the method proposed for such systems by Gendenshteïn in [14, 15]. The last section includes some final comments on the relation of this problem with that of the harmonic oscillator in the limit $\lambda \to 0$.

2 λ-dependent "Free Particle"

Let us first recall the case of the one-dimensional "free-particle" motion (in the sense that $\alpha = 0$) characterized by the Lagrangian

$$L(x, v_x, \lambda) = T_1(\lambda) = \frac{1}{2} \left(\frac{v_x^2}{1 + \lambda\,x^2} \right). \tag{3}$$

As it was remarked in [1], the Lagrangian function $T_1(\lambda)$ is invariant under the action of the vector field $X_x = X_x(\lambda)$ given by

$$X_x(\lambda) = \sqrt{1 + \lambda\,x^2}\;\frac{\partial}{\partial x}\,,$$

in the sense that we have

$$X_x^t(\lambda)\Big(T_1(\lambda)\Big) = 0\,,$$

where $X_x^t(\lambda)$ denotes the natural lift to the phase space $\mathbb{R}\times\mathbb{R}$ (tangent bundle in differential geometric terms) of the vector field $X_x(\lambda)$,

$$X_x^t(\lambda) = \sqrt{1 + \lambda\,x^2}\;\frac{\partial}{\partial x} + \Big(\frac{\lambda\,x\,v_x}{\sqrt{1 + \lambda\,x^2}}\Big)\frac{\partial}{\partial v_x}\,.$$

This vector field can be seen as a Killing vector field for the metric $g = (1 + \lambda\,x^2)^{-1}\,dx \otimes dx$, and generates a one-parameter group of isometries in this Riemann space. The natural measure in the real line is not invariant under such vector field; instead, the only invariant measures are the multiples of $d\mu = (1 + \lambda\,x^2)^{-1/2}\,dx$.

It is important to remark that in the space $L^2(\mathbb{R}, d\mu)$ of square integrable functions in the real line, the adjoint of the differential operator $\sqrt{1 + \lambda\,x^2}\,\partial/\partial x$ is precisely the opposite of such operator.

On the other side, with the momentum defined as usual,

$$p = \frac{\partial L}{\partial v_x} = \frac{v_x}{1 + \lambda\,x^2}\,,$$

the Legendre transformation $(x, v_x) \mapsto (x, p)$ leads to a Hamiltonian function given by

$$H = (1 + \lambda\,x^2)\frac{p^2}{2} = \frac{1}{2}\left(\sqrt{1 + \lambda\,x^2}\,p\right)^2\,.$$

Contrarily to the naive expectation of ordering ambiguities, however, the usual procedure of canonical quantization does not present any ambiguity because, as pointed out before, the linear operator (we put $\hbar = 1$)

$$P = -i\,\sqrt{1 + \lambda\,x^2}\,\frac{\partial}{\partial x}$$

is self adjoint in the space $L^2(\mathbb{R}, d\mu)$, and therefore the quantum Hamiltonian operator turns out to be

$$\widehat{H} = \frac{1}{2}\,P^2 = -\frac{1}{2}\left(\sqrt{1 + \lambda\,x^2}\,\frac{\partial}{\partial x}\right)^2 = -\frac{1}{2}\,(1 + \lambda\,x^2)\frac{\partial^2}{\partial x^2} - \frac{1}{2}\lambda\,x\,\frac{\partial}{\partial x}\,.$$

In presence of an interaction described by a potential $V_1(x)$ things works similarly and the Hamiltonian is then

$$\widehat{H}_1 = -\frac{1}{2}\,(1 + \lambda\,x^2)\frac{\partial^2}{\partial x^2} - \frac{1}{2}\,\lambda\,x\,\frac{\partial}{\partial x} + V_1(x)\,.$$

3   The factorization method

Hereafter, as the configuration space is one-dimensional, we will use the notation d/dx instead of the more traditional for the vector field $\partial/\partial x$.

Let us try to determine a function $W(x)$, called super-potential function, in such a way that the operator A and its adjoint operator $A^\dagger$, given by

$$
\begin{aligned}
A &= \frac{1}{\sqrt{2}} \left(\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + W(x) \right), \\
A^\dagger &= \frac{1}{\sqrt{2}} \left(-\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + W(x) \right),
\end{aligned}
$$

are such that $\widehat{H}_1 = A^\dagger A$ [13], i.e.

$$
\widehat{H}_1 = A^\dagger A = \frac{1}{2}\left[-\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + W(x) \right]\left[\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + W(x) \right]
$$

Therefore, the super-potential function W must satisfy the following Riccati type differential equation

$$
\sqrt{1 + \lambda\, x^2}\ W' - W^2 + 2\, V_1 = 0 .
$$

Once such a factorization is obtained, we can define a new quantum Hamiltonian operator

$$
\widehat{H}_2 = A\, A^\dagger = \frac{1}{2} \left[\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + W(x) \right] \left[-\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + W(x) \right],
$$

which is called the partner Hamiltonian. The new potential V_2 is given in terms of the super-potential W by

$$
V_2 = \frac{1}{2} \left(\sqrt{1 + \lambda\, x^2}\ W' + W^2 \right) .
$$

The operator A is such that $A\, \widehat{H}_1 = \widehat{H}_2\, A$ while $A^\dagger$ is such that $A^\dagger\, \widehat{H}_2 = \widehat{H}_1\, A^\dagger$. This shows that if $|\Psi\rangle$ is an eigenvector of $\widehat{H}_1$ corresponding to the eigenvalue E, then when $A|\Psi\rangle \neq 0$, $A|\Psi\rangle$ is an eigenvector of $\widehat{H}_2$ corresponding to the same eigenvalue, because

$$
\widehat{H}_2\, A\, |\Psi\rangle = A\, \widehat{H}_1\, |\Psi\rangle = E\, A\, |\Psi\rangle ,
$$

and similarly, when $|\Phi\rangle$ is an eigenvector of $\widehat{H}_2$ corresponding to the eigenvalue E and such that $A^\dagger\, |\Phi\rangle \neq 0$, then $A^\dagger\, |\Phi\rangle$ is an eigenvector of $\widehat{H}_1$ corresponding to the same eigenvalue E. In other words, the spectra of $\widehat{H}_1$ and $\widehat{H}_2$ are almost identical, the only difference appearing when either $|\Psi\rangle$ is an eigenvector of $\widehat{H}_1$ but $A\, |\Psi\rangle = 0$, or $|\Phi\rangle$ is an eigenvector of $\widehat{H}_2$ for which $A^\dagger\, |\Phi\rangle = 0$.

As an important first remark, the potential function V is only defined up to addition of a constant, and therefore all the preceding expressions can be extended to the case in which H is replaced by $H + c$, where c is a constant. As

4

another remark, it happens very often that some parameters may appear in the expression of the potential function, and therefore the super-potential function W will also depend of the values of such parameters. The most important case is when the explicit forms of the potential and its partner are quite similar and only differ in the values of the parameters, and then the problem is said to have shape invariance. As we will see this is the case for the quantum nonlinear oscillator we are considering in this paper.

4 The quantum nonlinear Oscillator

In the particular case of the nonlinear harmonic oscillator for which the Hamiltonian is given by

$$H = \frac{1}{2}\left[(1 + \lambda x^2)\, p_x^2 + \frac{\alpha^2\, x^2}{1 + \lambda\, x^2}\right]\ ,$$

the quantum Hamiltonian operator will be

$$\widehat{H}_1 = \frac{1}{2}\left[-(1 + \lambda x^2)\frac{d^2}{dx^2} - \lambda\, x\, \frac{d}{dx} + \frac{\alpha^2\, x^2}{1 + \lambda\, x^2}\right]\ . \tag{4}$$

Now, if for any real number β we define the linear operator in $L^2(\mathbb{R}, d\mu)$

$$A = \frac{1}{\sqrt{2}}\left(\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + \frac{\beta\, x}{\sqrt{1 + \lambda\, x^2}}\right)\ , \tag{5}$$

for which its adjoint operator is

$$A^\dagger = \frac{1}{\sqrt{2}}\left(-\sqrt{1 + \lambda\, x^2}\ \frac{d}{dx} + \frac{\beta\, x}{\sqrt{1 + \lambda\, x^2}}\right)\ , \tag{6}$$

then, we find that

$$A^\dagger A = -\frac{1}{2}\left(1 + \lambda\, x^2\right)\frac{d^2}{dx^2} - \frac{1}{2}\lambda\, x\, \frac{d}{dx} + \frac{1}{2}\beta\,(\beta + \lambda)\left(\frac{x^2}{1 + \lambda\, x^2}\right) - \frac{1}{2}\beta$$

where by simple comparing with (4) we conclude that the hamiltonian $\widehat{H}_1' = \widehat{H}_1 - (1/2)\beta$ admits a factorization

$$\widehat{H}_1' := A^\dagger\, A \tag{7}$$

whenever the parameters α and β are related by $\alpha^2 = \beta(\beta + \lambda)$, i.e.

$$\beta = -\frac{1}{2}\left(\lambda - \sqrt{\lambda^2 + 4\alpha^2}\right)\ .$$

Note that

$$\lim_{\lambda \to 0}\beta = \alpha\ .$$

Now for the partner hamiltonian $\widehat{H}_2' := A\,A^\dagger$ we find

$$\widehat{H}_2' = A\,A^\dagger = -\frac{1}{2}\left(1 + \lambda\,x^2\right)\frac{d^2}{dx^2} - \frac{1}{2}\,\lambda\,x\,\frac{d}{dx} + \frac{1}{2}\,\beta\left(\beta - \lambda\right)\left(\frac{x^2}{1 + \lambda\,x^2}\right) + \frac{1}{2}\,\beta \;. \tag{8}$$

Coming back to the general case, we recall that when a quantum Hamiltonian $\widehat{H}_1(\alpha)$ depending on a parameter α admits a factorization in such a way that the partner Hamiltonian $\widehat{H}_2(\alpha)$ is of the same form as $\widehat{H}_1(\alpha)$ but for a different value of the parameter α, it is usually said that there is shape invariance. More specifically, the quantum Hamiltonian $\widehat{H}_1(\alpha)$ admitting a factorization $\widehat{H}_1(\alpha) = A^\dagger(\alpha)\,A(\alpha)$ is said to be shape invariant when there exists a function f such that for the partner $\widehat{H}_2(\alpha) = A(\alpha)\,A^\dagger(\alpha)$ we have:

$$\widehat{H}_2(\alpha) = \widehat{H}_1(\alpha_1) + R(\alpha_1)\;, \tag{9}$$

where $\alpha_1 = f(\alpha)$ and $R(\alpha)$ is a constant depending on the parameter α. In this case, it was shown in [14] and [15] that there is a method for exactly computing all the spectrum of $\widehat{H}_1$ (see e.g. [9] for a modern approach). The bound state $|\Psi_0(\alpha)\rangle$ is found by solving $A(\alpha)|\Psi_0(\alpha)\rangle = 0$, and has a zero energy. Then, using (9) we can see that $|\Psi_0(\alpha_1)\rangle$ is an eigenstate of $\widehat{H}_2(\alpha)$ with an energy $E_1 = R(\alpha_1)$, because

$$\widehat{H}_2(\alpha)|\Psi_0(\alpha_1)\rangle = (\widehat{H}_1(\alpha_1) + R(\alpha_1))|\Psi_0(\alpha_1)\rangle = R(\alpha_1)|\Psi_0(\alpha_1)\rangle\;. \tag{10}$$

Then, $A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle$ is the first excited state of $\widehat{H}_1(\alpha)$, with an energy $E_1 = R(\alpha_1)$, because

$$\begin{aligned}
\widehat{H}_1(\alpha)A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle &= A^\dagger(\alpha)\widehat{H}_2(\alpha)|\Psi_0(\alpha_1)\rangle = A^\dagger(\alpha)(\widehat{H}_1(\alpha_1) + R(\alpha_1))|\Psi_0(\alpha_1)\rangle \\
&= R(\alpha_1)A^\dagger(\alpha)|\Psi_0(\alpha_1)\rangle\;.
\end{aligned}$$

This process should be iterated and we will find the sequence of energies for $\widehat{H}_1(\alpha)$

$$E_k = \sum_{j=1}^{k} R(\alpha_j), \qquad E_0 = 0\;, \tag{11}$$

the corresponding eigenfunctions being

$$|\Psi_n(\alpha_0)\rangle = A^\dagger(\alpha_0)A^\dagger(\alpha_1)\cdots A^\dagger(\alpha_{n-1})|\Psi_0(\alpha_n)\rangle\;, \tag{12}$$

where $\alpha_0 = \alpha$ and $\alpha_{j+1} = f(\alpha_j)$, namely, $\alpha_k = f^k(\alpha_0) = f^k(\alpha)$.

Now we can apply this process to the case we were considering, the parameter being β, because when comparing $\widehat{H}_1'$ given by (7) with its partner (8) we see that as

$$\widehat{H}_1'(\beta - \lambda) = -\frac{1}{2}\left[(1 + \lambda\,x^2)\frac{d^2}{dx^2} + \lambda\,x\,\frac{d}{dx}\right] + \frac{1}{2}\,(\beta - \lambda)\,\beta\left(\frac{x^2}{1 + \lambda\,x^2}\right) - \frac{1}{2}\,(\beta - \lambda)\;,$$

then

$$\widehat{H}_1'(\beta - \lambda) = \left[\widehat{H}_2'(\beta) - \left(\frac{1}{2}\right)\beta\right] - \frac{1}{2}\left(\beta - \lambda\right),$$

and, therefore,

$$\widehat{H}_2'(\beta) = \widehat{H}_1'(f(\beta)) + \beta - \frac{1}{2}\lambda,$$

where f is the function $f(\beta) = \beta - \lambda$. If R is the function defined by $R(\beta) = \beta + (1/2)\lambda$, then we see that

$$\widehat{H}_2'(\beta) = \widehat{H}_1'(\beta_1) + R(\beta_1).$$

Therefore, as the quantum nonlinear oscillator we are considering has a shape invariance, we can develop the method sketched before for finding both the spectrum and the corresponding eigenvectors.

The usual choice of parameters for these shape invariant systems is such that the function f corresponds to a displacement by one unit and then this suggests us to use the parameter $\gamma = \beta/\lambda$ instead of β.

5 The spectrum of the quantum nonlinear oscillator

Our starting point should be the bound state $|\Psi_0(\beta)\rangle$ of the Hamiltonian $\widehat{H}_1'$. This eigenvector is determined by the condition $A(\beta)|\Psi_0(\beta)\rangle = 0$. More specifically, we should solve the differential equation

$$\frac{d}{dx}\Psi_0 + \beta\left(\frac{x}{1 + \lambda x^2}\right)\Psi_0 = 0$$

and therefore the wave function of the fundamental state must be proportional to

$$\Psi_0(x) = \frac{1}{(1 + \lambda x^2)^{r_0}}, \qquad r_0 = \frac{\beta}{2\lambda}$$

The energies of the excited states will be

$$E_1' = R(\beta_1) = \beta - \lambda + \lambda/2 = \beta - \lambda/2,$$

and iterating the process we get

$$E_n' = \sum_{k=1}^n R(\beta_k) = \sum_{k=1}^n \left(\beta_k + \frac{\lambda}{2}\right) = \sum_{k=1}^n \left(\beta - \lambda k + \frac{\lambda}{2}\right),$$

and therefore,

$$E_n' = n\beta + \lambda\left[\frac{n}{2} - \sum_{k=1}^n k\right] = n\beta - \frac{n^2}{2}\lambda.$$

The energy of the eigenstates of $\widehat{H}_1 = \widehat{H}_1' + (1/2)\beta$ will be given by

$$E_n = n\,\beta - \frac{n^2}{2}\,\lambda + \frac{1}{2}\beta\,.$$

This relation can also be written as

$$E_n = -\frac{\lambda}{2}\left(n - \frac{\beta}{\lambda}\right)^2 + \frac{\beta\,(\beta - \lambda)}{2\,\lambda}\ .$$

The method developed in [14] and [15] also provides us the corresponding eigenfunctions as

$$\Psi_1 = A^\dagger(\beta)\Psi_0(\beta_1), \quad \ldots\ldots \quad , \quad \Psi_n = A^\dagger(\beta)A^\dagger(\beta_1)\cdots A^\dagger(\beta_{n-1})\Psi_0(\beta_n)\ .$$

There is a clear difference between the cases $\lambda > 0$ and $\lambda < 0$. In fact, let us first remark that the lowest value for E_n' is $E_0' = 0$. Therefore, if $\lambda > 0$ only those values n are allowed for which

$$\beta - \lambda\,\frac{n}{2} \geq 0 \Longrightarrow n \leq \frac{2\beta}{\lambda}\ ,.$$

On the contrary, when $\lambda < 0$ all natural numbers are allowed for n. It should also be remarked that as long as $\lambda \neq 0$, i.e, for both signs of λ, the eigenvalues are not equally spaced.

6   Final Comments and Outlook

In this paper we have analyzed a quantum version of a nonlinear quantum oscillator with an amplitude dependent angular frequency. In the limit when the parameter λ tends to zero we recover the usual harmonic oscillator. So, we first note that if $\lambda = 0$, then $\alpha = \beta$, and if we take into account that

$$\lim_{\lambda \to 0}(1 + \lambda\,x^2)^{\beta/(2\lambda)} = \exp\left(\lim_{\lambda \to 0}\frac{\beta}{2\lambda}\,(1 + \lambda\,x^2)\right) = e^{\frac{1}{2}\alpha x^2}\ ,$$

we see that the ground state we have found coincides with that of the corresponding quantum linear harmonic oscillator. The limit, when λ goes to 0, of operators A and $A^\dagger$ and the eigenvalues of the Hamiltonian are the corresponding annihilation and creation operators and the energy eigenvalues for the harmonic oscillator.

Another interesting issue concerns the behaviour of the position of the energy levels when the parameter λ changes. We remarked that equispacing of the levels, a characteristic property of the usual harmonic oscillator, no longer holds when $\lambda \neq 0$. In particular, the fundamental level is given by $E_0 = \beta/2$; while a superficial reading would suggest this is a constant independent of λ we must stress this is not the case, as β itself has been determined as a solution of the quadratic equation $\alpha^2 = \beta(\beta + \lambda)$, whose appropiate determination is

$$\beta = \frac{-\lambda + \sqrt{\lambda^2 + 4\alpha^2}}{2}$$

Thus, if we consider the nonlinear oscillator with fixed 'strenght' α and allow the nonlinear parameter to vary, the fundamental level will actually depend on λ. The fundamental level E_0 (for $n = 0$) has the standard oscillator value $\alpha/2$ for $\lambda = 0$, and is a positive and always decreasing function of λ. For large negative values of λ it approaches the large positive asymptotic regime $E_0(\lambda) \approx -\lambda/2$, while for large positive values of λ it tends to zero.

It is a well-known fact that canonical transformation and quantization do not commute. We have made a canonical point transformation in such a way that the new coordinate is the one rectifying the vector field generating the isometries of the metric associated to the Lagrangian describing the system and then the corresponding invariant measure is but the differential of such variable and the Hamiltonian becomes the square of the momentum, eliminating then order ambiguities. With this quantization procedure we have found that the spectrum and the corresponding eigenvectors can be easily found using the fact that the Hamiltonian admits such factorization that it is shape-invariant.

Acknowledgments.

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